Feb 27, 2017

Wednesday, March 1, 2017 10:27 PM

Finite Product of $(X_k, \mathcal{T}_k)$, $k=1,\dots,n$

For $P = \prod_{k=1}^{n} X_k$, the product topology is generated by $\mathcal{S} = \bigcup_{k=1}^{n} \{X_1 \times \cdots \times U_k \times \cdots \times X_n : U_k \in \mathcal{T}_k\}$.

After taking finite intersection, one has a base

$$\mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_n : U_k \in \mathcal{T}_k,\ k=1,\dots,n\}$$

Examples.

* $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R} \times \cdots \times \mathbb{R}$ (n-times)
* Annulus $= \{z \in \mathbb{C} : a \leq |z| \leq b\} \subset \mathbb{C} = \mathbb{R}^2$

  Cylinder $= \{(u,v,w) \in \mathbb{R}^3 : u^2+v^2=1,\ w \in [a,b]\} \subset \mathbb{R}^3$

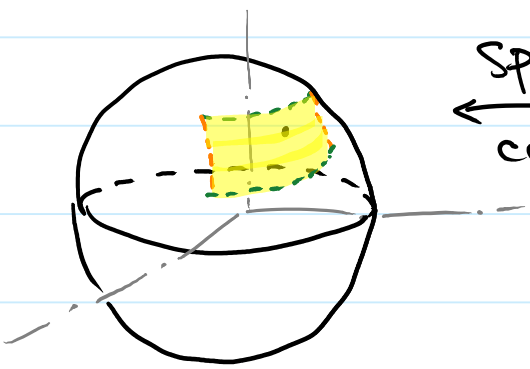
  $S^1 \times [a,b]$ where $S^1 = \{z \in \mathbb{C} : |z|=1\} \subset \mathbb{C} = \mathbb{R}^2$.

Non-examples Not product though local neighborhoods are of product form

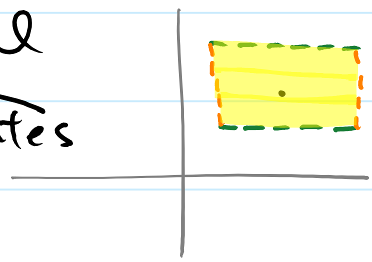
* Möbius strip
* Sphere

  $S^2 = \{(u,v,w) \in \mathbb{R}^3 : u^2+v^2+w^2=1\}$

  $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$



spherical coordinates



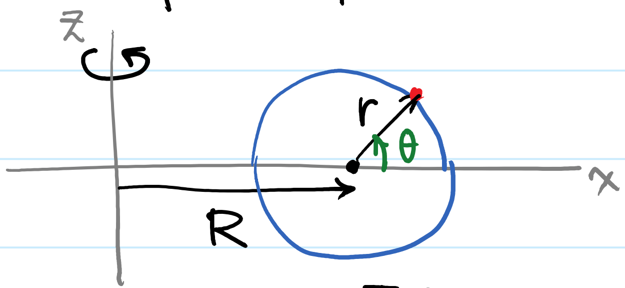
02-27-p2

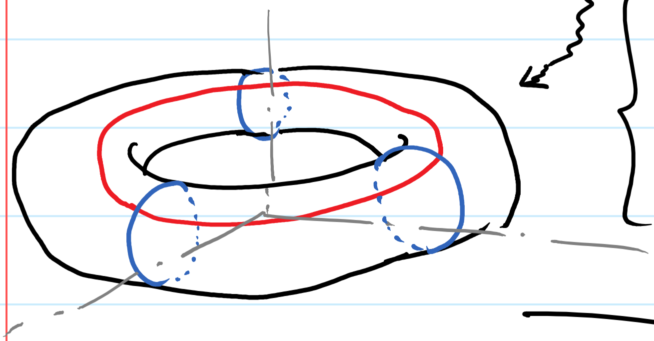
Wednesday, March 1, 2017 10:28 PM

Example.

* Torus, $\mathbb{T}$

As a surface of revolution





$$\left\{ \begin{array}{l} (u,v,w) \in \mathbb{R}^3 \\ u = (R + r\cos\theta)\cos\phi \\ v = (R + r\cos\theta)\sin\phi \\ w = r\sin\theta \end{array} \right\} \longrightarrow \begin{array}{c} S^1 \times S^1 \\ (e^{i\theta}, e^{i\phi}) \end{array}$$

In general, n-dimensional torus

$$\mathbb{T}^n = S^1 \times S^1 \times \cdots \times S^1 \quad (n\text{-times})$$

Let us first recall infinite product.

Definition. Given X_α, $\alpha \in I$.

We have $x \in \prod_{\alpha \in I} X_\alpha$ if

$$x : I \longrightarrow \bigcup_{\alpha \in I} X_\alpha \quad \text{such that } x(\alpha) \in X_\alpha$$

Examples

* $\mathbb{R}^n = \prod_{k \in n} X_k$, where $X_k = \mathbb{R}$, $n = \{0, 1, \cdots, n-1\}$

  Clearly, for $x \in \mathbb{R}^n$, $x : \{0, 1, \cdots, n-1\} \longrightarrow \mathbb{R}$
  $$k \longmapsto x(k) \in \mathbb{R}$$
  $$x = (x_0, x_1, \cdots, x_{n-1})$$

* For any I, if all $X_\alpha = Y$, then
  $$x \in \prod_{\alpha \in I} Y \quad \text{means} \quad x : I \longrightarrow Y$$
  Thus $\prod_{\alpha \in I} Y = Y^I$

* $I = \mathbb{N}$, $X_\alpha = \{0, 1\}$
  $$x \in \prod_{\alpha \in \mathbb{N}} \{0, 1\} = \{0, 1\}^{\mathbb{N}}$$ is an infinite sequence with entries $0, 1$

Recall in finite product, the generating set is $\bigcup_{k=1}^{n} \{ X_1 \times \cdots \times U_k \times \cdots \times X_n : U_k \in \mathcal{T}_k \}$

We would like to generalize this to infinite product, but it would be difficult to write.

Observe that
$$X_1 \times \cdots \times U_k \times \cdots \times X_n \xrightarrow{\pi_k} U_k$$
This will be useful in infinite product.

02-27-p4

Wednesday, March 1, 2017 10:29 PM

Definition. Given spaces $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in I$ and

$$\pi_\beta : \prod_{\alpha \in I} X_\alpha \longrightarrow X_\beta, \quad \pi_\beta(x) = x_\beta$$

For $P = \prod_{\alpha \in I} X_\alpha$, the product topology $\mathcal{T}_\Pi$ is generated by

$$\mathcal{S} = \{ \pi_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{T}_\alpha, \alpha \in I \}$$

After finite intersection, one got a base $\mathcal{B}$, which is hard to express, giving $\mathcal{T}_\Pi$

$$\mathcal{B} \cap \qquad\qquad \mathcal{T}_\Pi \subseteq$$

$$\mathcal{B}_{BOX} = \{ \prod_{\alpha \in I} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \}, \text{ which gives } \mathcal{T}_{BOX}$$

Example. Let $I = \mathbb{N}$, $X_\alpha = \{0,1\}$ discrete

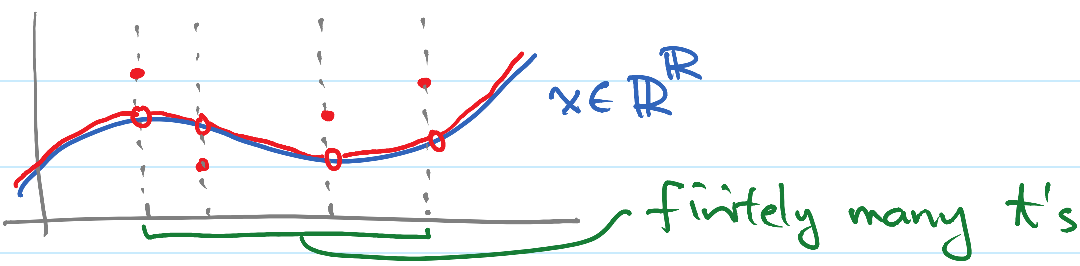
Consider $\bar{0} = (0,0,0,\cdots,0,\cdots) \in \{0,1\}^{\mathbb{N}}$.

A typical neighborhood of $\bar{0}$ in $\{0,1\}^{\mathbb{N}}$ is

$$\{0,1\} \times \cdots \times \underbrace{\{0\} \times \{0,1\} \times \cdots \times \{0\}}_{\text{finitely many}} \times \{0,1\} \times \cdots \times \cdots$$

x in a neighborhood of $\bar{0}$ means that x has finitely many of 0 entries.

* Let $I = \mathbb{R}$, $X_t = \mathbb{R}$ for $t \in I$, $\prod_{t \in \mathbb{R}} X_t = \mathbb{R}^{\mathbb{R}}$



Why use $\mathcal{T}_\Pi$ but not $\mathcal{T}_{BOX}$?

Consider all possible topologies on $P = \prod_{\alpha \in I} X_\alpha$,

$$\{\phi, P\} \subset \cdots\cdots \subset \mathcal{T}_\Pi \subset \cdots \subset \mathcal{T}_{BOX} \subset \cdots\cdots \subset \mathcal{P}(P)$$

and choose a topology $\mathcal{T}$ for P among them

The most natural functions on P are

$$\pi_\beta : (P, \mathcal{T}) \longrightarrow (X_\beta, \mathcal{T}_\beta)$$

To check its continuity, we need

$$\pi_\beta^{-1}(V) \in \mathcal{T} \quad \text{for each } V \in \mathcal{T}_\beta$$

If $\mathcal{T} = \mathcal{P}(P)$ then the above is always true

$\vdots \qquad \vdots$

If $\mathcal{T} = \{\phi, P\}$, then the above is not true

In fact, the above is true all the way from $\mathcal{P}(P)$ down to $\mathcal{T}_\Pi$ because by construction, $\mathcal{T}_\Pi$ is the smallest one containing all $\pi_\beta^{-1}(V)$ where $V \in \mathcal{T}_\beta$ and all $\beta \in I$.

Theorem. $\mathcal{T}_\Pi$ is the smallest topology for P to make each projection $\pi_\beta : P \longrightarrow X_\beta$ continuous.